Von Neumann algebras.

Definition. [Sakai] The von Neumann algebra A is a norm closed $*$-subalgebra of $B(H)$ which has a predual A_*, s.t. $A = (A_*)^*$

Facts.

- Every commutative von Neumann algebra is isomorphic to $L_\infty(X,\mu)$ with μ σ-finite.
  Its predual is $L_1(X,\mu)$.
  ($L_\infty(X,\mu)$ = classes of bounded measurable functions identified when equal almost everywhere,

the norm:

$$\|f\|_\infty = \inf \{ C \geq 0 \mid f(x) \leq C \text{ for almost every } x \}.$$

$L_1(X,\mu)$ = classes of measurable integrable functions identified when equal almost everywhere.

the norm: $\|f\|_1 = \int_X |f| \, d\mu$.

- Simple functions are dense in $L_1(X,\mu)$, so $L_1(X,\mu)$ is a "topological linearization" of the family of measurable subsets of finite measure.

They generalize elements of a set, whose linearization was a coalgebra.

Problem. Can a structure of a topological coalgebra be defined for preduals of noncommutative von Neumann algebras?

If the answer is in positive, they should be regarded as a "noncommutative" generalization of measure spaces.

Answear. Denoting by A_* the predual of a von Neumann algebra A we have

- Although the multiplication doesn't extend to the von Neumann-algebraic product $A \bar{\otimes} A \to A$ it does extend to the normal Haagerup product

$$m : A \overset{\sigma h}{\otimes} A \longrightarrow A$$

as a weak*-continuous completely contractive map.

• The domain has a natural predual,
the extended Haagerup product

$$A_* \overset{eh}{\otimes} A_* = CB^{\sigma}_{m}(A \times A, \mathbb{C})$$

and the predual map

$$m_* : A_* \longrightarrow A_* \overset{eh}{\otimes} A_*,$$

which is coassociative by associativity of m and

$$(A_* \overset{eh}{\otimes} A_*)^* = (A_*)^* \overset{\sigma h}{\otimes} (A_*)^* = A \overset{\sigma h}{\otimes} A \xrightarrow{m} A = (A_*)^*$$

- The predual $A_* \xrightarrow{\varepsilon} \mathbb{C}$ of the unit of A $\mathbb{C} \xrightarrow{\eta} A$ is the counit of the above comultiplication.

- The predual $B(H)_*$ is isomorphic to $T(H) \subset B(H)$, the ideal of <u>trace-class</u> operators.

Definition. Let $t \in B(H)$ and let (ψ_i) be a complete orthonormal system ("orthonormal basis" in the Hilbert space) in H. We say that t is <u>of trace class</u> if for $|t|^2 = t^* t$

$$|t|_1 := \sum_i \langle |t| \psi_i, \psi_i \rangle < \infty .$$

The set of trace-class operators is denoted by $T(H)$.

Definition. Let $t \in T(H)$ and (ψ_i) be an o.n. basis of H. Then one defines the <u>trace</u>

$$tr(t) := \sum_i \langle t\psi_i, \psi_i \rangle .$$

Facts about the trace.

- tr is independent of the choice of o.n. basis.
- $tr : T(H) \longrightarrow \mathbb{C}$ is linear
- $t \in T(H), b \in B(H) \Rightarrow tb, bt \in T(H)$ and

$$tr(tb) = tr(bt)$$

- $B(H) \longrightarrow T(H)^*, \; b \longmapsto (t \longmapsto tr(tb))$ is an isometric isomorphism.

Exercise 6. Show that for $b \in B(H)$

$$\|b\|^2 = \|b^*\|^2 = \|b^*b\| .$$

<u>Solution</u> Take $\psi \in H$ s.t. $\|\psi\| \leq 1$. Then

$$\|b\psi\|^2 = \langle b\psi, b\psi \rangle = \langle b^*b\psi, \psi \rangle \leq \|b^*b\psi\| \cdot \|\psi\|$$

$$\leq \|b^*b\| \cdot \|\psi\|^2 \leq \|b^*b\| \leq \|b^*\| \cdot \|b\| . \qquad (*)$$

$\Rightarrow \quad \|b\psi\| \cdot \|b\varphi\| \leq \|b^*\| \cdot \|b\| \quad$ if $\quad \|\psi\| \leq 1, \|\varphi\| \leq 1$.

$\Rightarrow \quad \|b\|^2 \leq \|b^*\| \cdot \|b\| \quad \Rightarrow \quad \|b\| \leq \|b^*\|$.

The same for b^* instead of b, so $\quad \|b^*\| \leq \|b\|$.

$\Rightarrow \quad \|b\| = \|b^*\|. \quad \left({*\atop *}\right)$

Then by $(*)$ and $\binom{*}{*}$

$$\|b\|^2 \overset{(*)}{\leq} \|b^*b\| \overset{\binom{*}{*}}{\leq} \|b^*\|\cdot\|b\| = \|b\|^2$$

$\Rightarrow \quad \|b\|^2 = \|b^*b\|$. $\square$

Exercise 6. Show that for $b \in B(H)$ $|b| \in B(H)$ and for $t \in T(H)$ $|t|_1$ is independent of the choice of orthonormal basis.

Solution. $\| |b| \psi \|^2 = \langle |b|\psi, |b|\psi \rangle = \langle |b|^2 \psi, \psi \rangle = \langle b^* b \psi, \psi \rangle$

$= \langle b\psi, b\psi \rangle = \| b\psi \|^2 \leq \|b\|^2 \cdot \|\psi\|^2.$

Now, $|t| = \sqrt{|t|}\sqrt{|t|}$, $\sqrt{|t|}^* = \sqrt{|t|}$.

Let (φ_i) be another o.n. basis. Then

$\sum_i \langle |t|\psi_i, \psi_i \rangle = \sum_i \langle \sqrt{|t|}\psi_i, \sqrt{|t|}\psi_i \rangle$

$$= \sum_i \sum_j |\langle \sqrt{|t|}\psi_i, \varphi_j \rangle|^2$$

$$= \sum_i \sum_j |\langle \psi_i, \sqrt{|t|}\varphi_j \rangle|^2$$

$$= \sum_j \sum_i |\langle \sqrt{|t|}\varphi_j, \psi_i \rangle|^2 = \sum_j \langle |t|\varphi_j, \varphi_j \rangle . \quad \square$$

Definition. An operator $S \in B(H)$ is called Hilbert-Schmidt if $\exists$ an o.n. basis (ψ_i)

$$\|S\|_{HS}^2 := \sum_i \|S\psi_i\|^2 < \infty .$$

The set of such operators is denoted by $HS(H)$.

Exercise 7. Show that for any $b \in B(H)$ and o.n. bases (φ_i), (ψ_j) of H we have

$$\sum_j \|b\psi_j\|^2 = \sum_i \|b^*\varphi_i\|^2$$

Solution. Use the Parseval identity

$$b\psi_j = \sum_i \langle \varphi_i, b\psi_j \rangle \varphi_i \quad , \quad \|b\psi_j\|^2 = \sum_i |\langle \varphi_i, b\psi_j \rangle|^2$$

$$\Rightarrow \quad \sum_j \|b\psi_j\|^2 = \sum_j \sum_i |\langle \varphi_i, b\psi_j \rangle|^2 = \sum_i \sum_j |\langle \varphi_i, b\psi_j \rangle|^2$$

$$= \sum_i \sum_j |\langle b^*\varphi_i, \psi_j \rangle|^2 = \sum_i \sum_j |\langle \psi_j, b^*\varphi_i \rangle|^2$$

$$= \sum_i \|b^*\varphi_i\|^2 . \qquad \square$$

This implies that $\|S\|_{HS}$ is independent of the choice of o.n. basis and

$$S \in HS(H) \iff S^* \in HS(H)$$

as follows

$$(\psi_j) \sim (\varphi_i) \Rightarrow \sum_i \|S\varphi_i\|^2 = \sum_i \|S^*\varphi_i\|^2 = \sum_j \|S\psi_j\|^2$$

$\Rightarrow$ $\|S\|_{HS}$ independent of the choice of o.n. basis and

$$\|S\|_{HS} = \|S^*\|_{HS} .$$

Exercise 8. Show that $HS(H)$ is an ideal in $B(H)$

Solution. $\|bs\|_{HS}^2 = \sum_i \|bs\psi_i\|^2 \leq \sum_i \|b\|^2 \|s\psi_i\|^2 = \|b\|^2 \|s\|_{HS}^2 < \infty$

$\|sb\|_{HS} = \|(sb)^*\|_{HS} = \|b^* s^*\|_{HS} \leq \|b^*\| \, \|s^*\|_{HS} = \|b\| \cdot \|s\|_{HS} < \infty$. $\square$

Together with Exc. 7 this means that $HS(H)$ is a $*$-ideal in $B(H)$.

Proposition. TFAE

- $t \in T(H)$

- $\exists\, s_1, s_2 \in HS(H) \quad t = s_1 s_2$.

Proof. If $t \in T(H)$

$$\Longrightarrow \quad \| |t|^{1/2} \|_{HS}^2 = \sum_i \langle |t|^{1/2} \psi_i, |t|^{1/2} \psi_i \rangle = \sum_i \langle \psi_i, |t| \psi_i \rangle < \infty$$

$$\Longrightarrow \quad |t|^{1/2} \in HS(H).$$

If $|b|^{1/2} \in HS(H)$

$\Longrightarrow \quad b = U|b| = (U|b|^{1/2})(|b|^{1/2})$. Hence $HS(H)$ is an ideal

$b = s_1 s_2$, where $s_1 = U|b|^{1/2}$, $s_2 = |b|^{1/2} \in HS(H)$.

polar decomposition

Suppose $b = s_1 s_2$, $s_1, s_2 \in HS(H)$. Take the polar decomposition $b = U|b|$. Then $U^*U|b| = |b|$

But $U|b| = s_1 s_2 \Rightarrow |b| = U^*U|b| = U^* s_1 s_2 = (U^* s_1) s_2$

$HS(H)$ ideal $\Rightarrow U^* s_1 \in HS(H) \Rightarrow |b| \in HS(H)$.

Suppose $|b| = s_1 s_2$, $s_1, s_2 \in HS(H)$. Then $b \in HS(H)$

$\Rightarrow b^* \in HS(H)$

$$\Rightarrow \quad \sum_i \langle \psi_i, |b| \psi_i \rangle = \sum_i \langle \psi_i, s_1 s_2 \psi_i \rangle = \sum_i \langle s_1^* \psi_i, s_2 \psi_i \rangle$$

$$\leq \sum_i \|s_1^* \psi_i\| \cdot \|s_2 \psi_i\| \leq \Big(\sum_i \|s_1^* \psi_i\|^2\Big)^{1/2} \Big(\sum_i \|s_2 \psi_i\|^2\Big)^{1/2}$$

$$= \|s_1^*\|_{HS} \cdot \|s_2\|_{HS} < \infty$$

Cauchy-Schwartz

$\Rightarrow b \in T(H)$. $\square$

Exercise 9. Show that $T(H) \subset B(H)$ is an ideal closed under $*$.

<u>Solution.</u> $t \in T(H)$, $b \in B(H)$

$$|bt|_1 \leq \|b\| \cdot |t|_1 \quad , \quad |tb|_1 \leq |t|_1 \|b\| .$$

$$t = s_1 s_2 \quad , s_1, s_2 \in HS(H) \implies t^* = s_2^* s_1^* \quad , \quad s_2^*, s_1^* \in HS(H).$$

$\blacksquare$

Corollary. A noncommutative measure space C is a topological coalgebra $\Delta : C \to C \overset{eh}{\otimes} C$, $\varepsilon : C \to \mathbb{C}$ whose continuous dual C^* embeds into $B(H)$ as a $*$-stable weak* closed subalgebra.

But there is a better, fully coalgebraic description of noncommutative measure spaces thanks to the following theorem. Here we regard A_* as a subspace of A^*.

Theorem. [Effros-Ruan] Let $A \subset B(H)$ be a von Neumann algebra. We have a complete isometry

$$A_* = T(H)/A_\perp .$$

Proof. [Sketch of] Since the restriction map

$$B(H)_* = T(H) \longrightarrow A_*, \quad t \longmapsto t|_A$$

is a complete contraction with kernel $A_\perp$, it induces a complete contraction

$$\pi : T(H)/A_\perp \longrightarrow A_* .$$

On the other hand, let us take

$$\varphi \in M_n(A_*) = CB^\sigma(A, M_n(\mathbb{C})).$$

By von Neumann algebra theory every normal completely bounded map has a norm preserving normal completely bounded extension

$$\tilde{\varphi} \in M_n(T(H)) = CB^\sigma(B(H), M_n(\mathbb{C})).$$

Therefore π must be a completely isometric isomorphism. $\square$

Corollary. A noncommutative measurable space C is a topological coalgebra $\Delta : C \to C \overset{\epsilon h}{\otimes} C$, $\varepsilon : C \to \mathbb{C}$ which is a $*$-quotient of the topological $*$-coalgebra $T(H)$ of trace class operators.

The next task will be to extend the notion of measure spaces to the noncommutative setting.